A Probabilistic Approach to the Homogenization of Divergence-Form Operators in Periodic Media

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Abstract: We prove here using stochastic analysis the homogenization property of second-order divergence-form operators with lower-order differential terms (possibly highly-oscillating) in periodic media. The coefficients are not assumed to have any regularity, so the stochastic calculus theory for processes associated to Dirichlet forms is used. The Girsanov Theorem and the Feynman-Kac formula are used to work on the probabilistic representation of the solutions of some PDEs.

Keywords: divergence-form operators, Dirichlet forms, homogenization, Feynman-Kac formula, Girsanov Theorem

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1 Introduction

The homogenization property for divergence-form operators with discontinuous coefficients is proved using stochastic calculus for processes generated by Dirichlet forms. As a sequel, we directly obtain pointwise convergence of the solutions of the corresponding PDEs, instead of functional convergence. Afterwards, we extend the method used by É. Pardoux in [16] for a highly-oscillating zero-order term to deal with a differential term of the form $\partial_{x_i}(\widehat{b}(\cdot/\varepsilon)\cdot)$ when $\widehat{b}$ is not regular.

Instead of considering second-order non-divergence form operators (linked to Stochastic Differential Equations (SDE)) we deal with a family of divergence-form operators

$$L^\varepsilon = \frac{1}{2}\frac{\partial}{\partial x_i}\left(a_{i,j}(\cdot/\varepsilon)\frac{\partial}{\partial x_j}\right) + \left(\frac{1}{\varepsilon}b_i(\cdot/\varepsilon) + c_i(\cdot/\varepsilon)\right)\frac{\partial}{\partial x_i}, \tag{1}$$

where the coefficients a, b and c are periodic. Here, $a(x) = (a_{i,j}(x))_{i,j=1}^N$ is a uniformly elliptic, symmetric matrix, and the vector valued functions $b(x) = (b_i(x))_{i=1}^N$ and $c(x) = (c_i(x))_{i=1}^N$ are bounded. No assumption of regularity on the coefficients is required.

In fact, L^ε is the infinitesimal generator of a continuous, conservative, Feller process X^ε. Also an homogenization result on L^ε is equivalent to the convergence in distribution of X^ε to a non-standard Brownian Motion. However with a divergence-form operator with irregular coefficients, the process X^ε is not in general the solution of some SDE: if it was, then this SDE would have a drift term including $\int_0^t \partial_{x_i} a_{i,j}(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s$, which is not defined for irregular coefficients a. So we have to use the stochastic calculus for processes generated by Dirichlet forms (a Dirichlet form being the bilinear form associated to L^ε) as it is mainly developed in [5]. Although we follow the general method first developed by M. Freidlin [6] and widely used since (see [2] for example), we have to be careful, because the theory of Dirichlet forms is naturally developed in the framework of functional spaces, here we must use the L^2 and Sobolev spaces. This leads to a behaviour slightly different from that of the usual Itô calculus. For example, a formula may not be defined under the distribution of the process starting from a fixed point. Also, the gradient of the correctors (that is the functions that transform X^ε into a martingale) are not bounded. For example, when dealing with the highly oscillating zero-order term, a supplementary argument is required with respect to the one given in [16] (see Proposition 3), where everything is bounded.

One may wonder if instead of using the context of Dirichlet form, we could simply proceed by approximations. However, this leads to problems of interchanging limits.

This article is organized as follows: in Section 2, we recall the main properties of a process generated by a divergence-form operator. In Section 3, the convergence of the stochastic processes X^ε is proved. Section 4 is devoted to the proof of the homogenization property with differential terms of order zero, using the Feynman-Kac formula for representing the semi-group.

2 Divergence-form operator and Markov process

Let $\mathcal{O}$ be a connected, open subset of $\mathbb{R}^N$. Through this Section, let L be the operator

$$L = \frac{1}{2}\frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial}{\partial x_j}\right) + b_i\frac{\partial}{\partial x_i},\ \mathrm{D}(L) = \left\{\, f \in \mathrm{H}^1_0(\mathcal{O}) \,\middle|\, Lf \in \mathrm{L}^2(\mathcal{O}) \,\right\},$$

where $a(x) = (a_{i,j}(x))_{i,j=1}^N$ is a measurable function in the space of symmetric $N \times N$-matrices for which there exist some positive constants λ and Λ such that

$$\lambda|\xi|^2 \leqslant a_{i,j}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2\ \forall\xi \in \mathbb{R}^N,\ \forall x \in \mathcal{O},$$

and b is a measurable function with value in $\mathbb{R}^N$. The norm of $b(x)$ is assumed to be bounded by Λ.

It is well-known that the operator $(L, \mathrm{D}(L))$ is the infinitesimal generator of a semi-group $(P_t)_{t>0}$ and of the resolvent $(G_\alpha)_{\alpha>\alpha_0}$ defined for any α greater than α_0 depending only on λ and Λ. For any function f in $\mathrm{L}^2(\mathcal{O})$, $P_t f$ is the weak solution of the parabolic PDE

$$\partial_t u(t,x) = Lu(t,x) \text{ with } u(0,x) = f(x) \text{ and } u(t,\cdot) \in \mathrm{H}^1_0(\mathcal{O})\ \forall(t,x) \in \mathbb{R}_+^* \times \mathcal{O},$$

and $G_\alpha f$ is the weak solution of the elliptic PDE

$$(\alpha - L)v = f,\ \text{where } v \in \mathrm{H}^1_0(\mathcal{O}).$$

Furthermore, there exists a transition density function p such that $x \mapsto \int_{\mathcal{O}} p(t,x,y)f(y)\,\mathrm{d}y$ is a version of $x \mapsto P_t f(x)$ for any function f in $\mathrm{L}^2(\mathcal{O})$. From standard results in PDE theory, this function p satisfies the Aronson estimate [1, 19]

$$p(t,x,y) \leqslant \frac{M}{t^{N/2}} e^{-|x-y|^2/Mt}, \forall t > 0, \tag{2}$$

and the Giorgi-Nash estimate [11, 19]

$$|p(t',x',y') - p(t,x,y)| \leqslant \frac{C}{\delta^N}\left(\frac{\sqrt{|t'-t|} \vee |x'-x| \vee |y'-y|}{\delta}\right)^\alpha \tag{3}$$

for any $(t',x',y'),(t,x,y)\in[\delta^2,+\infty)\times\mathcal{O}\times\mathcal{O}$ with $|y'-y|\vee|x'-x|\leqslant\delta$. The constants M, C and α in (2) and (3) depends only on λ and Λ.

Hence, it is easily shown that the semi-group $(P_t)_{t>0}$ is a Feller semi-group, and is the generator of a Hunt process $(\mathsf{X},\mathbb{P}_x)_{x\in\mathcal{O}}$ which is continuous up to is life-time.

The key tool to study this process is the theory of Dirichlet form [5, 13], since the bilinear form

$$\mathcal{E}(u,v)=\frac{1}{2}\int_{\mathcal{O}}a_{i,j}\partial_{x_i}u\partial_{x_j}v\,\mathrm{d}x-\int_{\mathcal{O}}b_i\partial_{x_i}uv\,\mathrm{d}x,\ \mathrm{D}(\mathcal{E})=\mathrm{H}^1_0(\mathcal{O}),$$

is a Dirichlet form on $\mathrm{L}^2(\mathcal{O})$. Lot of results are more natural for symmetric Dirichlet forms, — that is with $b=0$. But in fact, the distribution starting at a given point of the process generated by $\mathcal{E}$ is deduced from that of the same Dirichlet form with $b=0$ by a Girsanov transform, as with non-divergence-form operators. We have to note that this property is in fact true for *any* starting point [12, 3, 10].

If we consider a bounded domain $\mathcal{O}$ and choose the domain of the Dirichlet form to be $\mathrm{H}^1_0(\mathcal{O})$, — which is related to the Dirichlet problem, — then process generated by this Dirichlet form is related to a conservative generated by a Dirichlet form whose coefficients extend those of the previous one. Using Theorem 4.2.2 and Lemma 2.3.4 in [5, p. 154 and p. 95], if the coefficient a is extended to a function on $\mathbb{R}^N$ satisfying the ellipticity and boundedness condition, the process distribution of the process X is equal to the distribution of the conservative process whose infinitesimal generator is $\frac{1}{2}\partial_{x_i}(a_{i,j}\partial_{x_j})$ defined on $\mathbb{R}^N$ and killed when it exits from $\mathcal{O}$. This result may be carried for non-symmetric operator with the help of the Girsanov transform.

Let $\tau(\omega)=\inf\{\,t>0\,|\,\omega_t\notin\mathcal{O}\,\}$ is the first exit-time from $\mathcal{O}$ for any continuous function ω on $[0,+\infty)$ with value in $\mathbb{R}^N$. If the boundary of $\mathcal{O}$ is smooth enough, then $\tau(\mathsf{X}^\varepsilon)$ converges in distribution to $\tau(\mathsf{X})$, where X^ε are processes generated by divergence-form operators converging in distribution to X.

3 Homogenization: convergence of the stochastic processes

We assume in this section that $\mathcal{O}=\mathbb{R}^N$. Let c be a measurable function with value in $\mathbb{R}^N$ also bounded by Λ. Now, let us assume that the coefficients a, b and c are periodic. Let us denote by $\square$ the unit torus of $\mathbb{R}^N$: $\square=\mathbb{R}^N/\mathbb{Z}^N$. The process X^ε is the one generated by the operator L^ε given by (1).

Proposition 1 (Homogenization result). *There exist some constant β (see* (8) *below) and some constant coefficients $\overline{a} = (\overline{a}_{i,j})_{i,j=1}^N$ and $\overline{c} = (\overline{c}_i)_{i=1}^N$ such that the process $\mathsf{X}_t^\varepsilon - \beta t/\varepsilon$ converges in distribution to the process generated by $\overline{L} = \frac{1}{2}\overline{a}_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \overline{c}_i \frac{\partial}{\partial x_i}$.*

Let $\mathrm{L}^2(\Box)$ and $\mathrm{H}^1(\Box)$ be the space of functions locally in $\mathrm{L}^2(\mathbb{R}^N)$ and $\mathrm{H}^1(\mathbb{R}^N)$ which are $\Box$-periodic. The injection from the space $\mathrm{H}^1(\Box)$ to $\mathrm{L}^2(\Box)$ is compact. Furthermore, if m is a $\Box$-periodic function such that $\int_\Box m(x)\,\mathrm{d}x = 1$ and $0 < \kappa \leqslant m(x) \leqslant \delta$, $\forall x \in \Box$ for some constants κ and δ, then there exists a constant C such that

$$\left\| f - \int_\Box f(x) m(x)\,\mathrm{d}x \right\|_{\mathrm{L}^2(\Box)} \leqslant C \left\|\nabla f\right\|_{\mathrm{L}^2(\Box)^N} \tag{4}$$

for any f in $\mathrm{H}^1(\Box)$. The inequality (4) is the *Poincaré inequality* and leads to the specificity of the homogenization property in periodic media, in opposite to a general random media.

In a first time, we want to define a Markov process corresponding to the projection on the torus of the process generated by the divergence form operator L having periodic coefficients. Thanks to the compactness of the torus, this process will be ergodic.

Let q be the function from $\mathbb{R}_+^* \times \Box \times \Box$ to $\mathbb{R}_+^*$ defined by $q(t,x,y) = \sum_{k\in\mathbb{Z}^d} p(t,x,y+k)$. Using the Aronson estimate, it is clear that there exist some constants $\kappa = \kappa(\lambda,\Lambda,n)$ and $\delta = \delta(\lambda,\Lambda,n)$ such that $\delta \leqslant q(1,x,y) \leqslant \kappa$. The family of operators $(Q_t)_{t\geqslant 0}$ defined by $Q_t f(x) = \int_\Box q(t,x,y) f(y)\,\mathrm{d}y$ defined a semi-group on $\mathrm{L}^p(\Box)$ and any $1 \leqslant p \leqslant \infty$. The infinitesimal generator of $(Q_t)_{t>0}$ is

$$\widetilde{L} = \frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial}{\partial x_j}\right) + b_i \frac{\partial}{\partial x_i} \text{ and } \mathrm{D}(\widetilde{L}) = \{\, f \in \mathrm{H}^1(\Box) \mid \widetilde{L} f \in \mathrm{L}^2(\Box)\,\}.$$

It follows from the compact injection of $\mathrm{H}^1(\Box)$ into $\mathrm{L}^2(\Box)$ that the resolvent $\widetilde{G}_\alpha = (\alpha - \widetilde{L})^{-1}$ of $\widetilde{L}$ is clearly a compact operator.

Using the maximum principle and the periodicity of the functions in $\mathrm{L}^2(\Box)$, any solution of $\widetilde{L}u = 0$ is constant. Thanks to the Fredholm alternative, there exists a unique solution to $\widetilde{L}^* m = 0$ such that $\int_\Box m(x)\,\mathrm{d}x = 1$. As in [7, p. 328], m is non-negative. With the Harnack principle [18, Theorem 8.1, p. 237] and a regularity result [18, Theorem 7.1, p. 232], m is in fact positive and continuous, m is the unique invariant probability m for Q_t. Furthermore, $\delta(\lambda,\Lambda,n) \leqslant m(x) \leqslant \kappa(\lambda,\Lambda,n)$ for any $x \in \Box$.

Using a density argument, it may be proved that for any function $u \in \mathrm{D}(\widetilde{L})$,

$$\int_\Box \widetilde{L}u(x)u(x)m(x)\,\mathrm{d}x = \frac{-1}{2}\int_\Box m(x)\cdot a_{i,j}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j}\,\mathrm{d}x. \tag{5}$$

We deduce then that $\widetilde{L}$ is a closed operator with a dense domain in $\mathrm{L}^2(\square)$. As a sequel, we deduce that for any f in $\mathrm{H}^{-1}(\square)$ such that $\langle f, m\rangle_{\mathrm{H}^{-1}(\square),\mathrm{H}^1(\square)} = 0$, there exists a unique solution $u \in \mathrm{H}^1(\square)$ up to some additive constant to

$$\widetilde{\mathcal{E}}(u,v) = -\langle f, v\rangle_{\mathrm{H}^{-1}(\square),\mathrm{H}^1(\square)}, \quad \forall v \in \mathrm{H}^1(\square),$$

where $\widetilde{\mathcal{E}}$ is the bilinear form associated to $\widetilde{L}$ by the relation $\widetilde{\mathcal{E}}(u,v) = -\langle \widetilde{L}u, v\rangle$ for $(u,v) \in \mathrm{D}(\widetilde{L}) \times \mathrm{H}^1(\square)$.

It is now time to add our first-order term $\varepsilon c\nabla$. Let ${}^{\varepsilon}L$ be the operator

$${}^{\varepsilon}L = L + \varepsilon c_i \frac{\partial}{\partial x_i},\ \mathrm{D}({}^{\varepsilon}L) = \mathrm{D}(L),$$

which is the infinitesimal generator of a strong Markov process ${}^{\varepsilon}\mathsf{X}$ with value in $\mathbb{R}^N$. The operator corresponding to ${}^{\varepsilon}L$ on the space of periodic function will be ${}^{\varepsilon}\widetilde{L} = \widetilde{L} + \varepsilon c_i \partial_{x_i}$, and generates a semi-group $({}^{\varepsilon}Q_t)_{t>0}$.

We are now concerned with the convergence in distribution of $\mathsf{X}^{\varepsilon} \overset{\mathrm{dist}}{=} \varepsilon \cdot {}^{\varepsilon}\mathsf{X}_{\cdot/\varepsilon^2}$.

A variation of the spectral gap inequality leads to establishing the following convergence result. Here, the version of the semi-group is the one given using the density transition function. This Proposition have to be proved for functions in $\mathrm{L}^1(\square)$, since it will be applied on some functions involving the gradient of the correctors belonging only to $\mathrm{L}^1(\square)$. In fact, we give a statement slightly stronger, whose use will be justified in Section 4.2.

Proposition 2. *Let $(f_n)_{n\in\mathbb{N}}$ be a family of functions in $\mathrm{L}^1(\square)$ converging in $\mathrm{L}^1(\square)$ to f. Let $(n_\varepsilon)_{\varepsilon>0}$ be a sequence of integers increasing to infinity as ε decreases to 0. Then there exist positive constants K, ρ and C depending only on λ, Λ and N such that*

$$\sup_{x\in\square} \left| {}^{\varepsilon}Q_t f_{n_\varepsilon}(x) - \int_\square f(x)m(x)\,\mathrm{d}x \right| \leqslant K \left\| f \right\|_{\mathrm{L}^1(\square)} e^{-C(\varepsilon)t} + \left\| f_{n_\varepsilon} - f \right\|_{\mathrm{L}^1(\square)} \quad (6)$$

where $C(\varepsilon)$ converges to ρ as ε goes to 0.

Now, we have to recall that for a periodic function f, ${}^{\varepsilon}Q_t f(x) = \mathbb{E}_x\left[\, f({}^{\varepsilon}\mathsf{X}_t)\,\right]$. Following the argument exposed in [2, p. 338], we deduce from (6) the following convergence result, which has the flavor of an ergodic theorem.

Corollary 1. *Under the hypotheses of Proposition 2,*

$$\sup_{x\in\mathbb{R}^N} \mathbb{E}_x\left[\left| \varepsilon^2 \int_1^{t/\varepsilon^2} f_{n_\varepsilon}({}^{\varepsilon}\mathsf{X}_s)\,\mathrm{d}s - t\int_\square f(y)m(y)\,\mathrm{d}y \right|^2 \right] \xrightarrow[\varepsilon\to 0]{} 0, \quad (7)$$

where f and f_{n_ε} are seen as periodic functions on $\mathbb{R}^N$.

We note that since we are working with functions only in $\mathrm{L}^1(\square)$, we cannot be ensured that $\int_0^t f({}^\varepsilon\mathsf{X}_s)\,\mathrm{d}s$ is finite under $\mathbb{P}_x$ for a given starting point. This is why we start at time 1.

Let $\beta = (\beta_i)_{i=1,\dots,N}$ be the vector defined by

$$\beta_i = \frac{1}{2}\sum_{j=1}^N \int_\square a_{i,j}(x)\frac{\partial m(x)}{\partial x_i}\,\mathrm{d}x + \int_\square b_i(x)m(x)\,\mathrm{d}x. \tag{8}$$

Remark. If $b = 0$, then $m(x) = 1$ and automatically, $\beta = 0$.

For $i = 1,\dots,N$, let w_i be the unique 1-periodic solution in $\mathrm{H}^1(\square)$ of the problem

$$\begin{cases} \widetilde{L}w_i = -\dfrac{1}{2}\sum_{j=1}^d \dfrac{\partial a_{i,j}}{\partial x_j} - b_i + \beta_i, \\ \displaystyle\int_\square w_i(x)m(x)\,\mathrm{d}x = 0. \end{cases}$$

According to Theorem 7.2 in [18, p. 232], each function $\widetilde{w}_i$ is continuous, and then bounded. We set

$$\widetilde{w}_i(x) = x + w_i(x) \text{ for } i = 1,\dots,N,$$

where w_i is extended to a periodic function on $\mathbb{R}^N$. If $\widetilde{w}_i^\varepsilon = \varepsilon\widetilde{w}_i(\cdot/\varepsilon)$, then the functions $\widetilde{w}_i$ and $\widetilde{w}_i^\varepsilon$ satisfy (in the weak sense)

$$\mathrm{div}(a\nabla\widetilde{w}_i) + b\nabla\widetilde{w}_i = \beta_i \text{ and } \mathrm{div}(a(x/\varepsilon)\nabla\widetilde{w}_i^\varepsilon(x)) + \frac{1}{\varepsilon}b(x/\varepsilon)\nabla\widetilde{w}_i^\varepsilon(x) = \frac{\beta_i}{\varepsilon}$$

So, with the Itô-Fukushima decomposition (see [5]) and the help of the Girsanov transform to deal with the first-order term, for $u = 1,\dots,m$,

$$\widetilde{w}_i^\varepsilon(\mathsf{X}_t^\varepsilon) - \widetilde{w}_i(\mathsf{X}_{\varepsilon^2}^\varepsilon) = \mathsf{M}_t^{\varepsilon,i} + \int_{\varepsilon^2}^t \langle c, \nabla\widetilde{w}_i\rangle(\mathsf{X}_s^\varepsilon/\varepsilon)\,\mathrm{d}s + \frac{\beta}{\varepsilon}t,\ \mathbb{P}_x\text{-a.e.}, \tag{9}$$

for any $x \in \mathbb{R}^N$, where $\mathsf{M}_t^\varepsilon = (\mathsf{M}_t^{\varepsilon,1},\dots,\mathsf{M}_t^{\varepsilon,N})$ is a local martingale with cross-variations

$$\begin{aligned}\langle \mathsf{M}^{\varepsilon,i}, \mathsf{M}^{\varepsilon,j}\rangle_t &= \int_0^t \langle a\nabla\widetilde{w}_i, \nabla\widetilde{w}_j\rangle(\mathsf{X}_s^\varepsilon/\varepsilon)\,\mathrm{d}s \overset{\text{dist}}{=} \varepsilon^2\int_0^{t/\varepsilon^2} \langle a\nabla\widetilde{w}_i, \nabla\widetilde{w}_j\rangle({}^\varepsilon\mathsf{X}_s/\varepsilon)\,\mathrm{d}s \\ &\xrightarrow[\varepsilon\to 0]{} t\int_\square \langle a\nabla\widetilde{w}_i, \nabla\widetilde{w}_j\rangle(x)m(x)\,\mathrm{d}x = t\overline{a}_{i,j}. \end{aligned} \tag{10}$$

We start at time ε^2 in (9) to ensure that this relation is true for any starting point, and not only for quasi-every starting point (see [5, 13]).

Using the boundedness of $w(x)$, the right-hand side term in (9) will have the same limit as $\mathsf{X}_t^\varepsilon - \mathsf{X}_{\varepsilon^2}^\varepsilon$. The Central Limit Theorem on martingales (see *e.g.*, Theorem 7.1.4 in [4, p. 339]) implies the existence of a martingale $\overline{M}$ whose cross-variations $\langle \overline{M}^i, \overline{M}^j \rangle_t$ are $t\overline{a}_{i,j}$ for $i, j = 1, \dots, N$, and such that M^ε converges in distribution to $\overline{M}$ in the space of continuous functions.

With (7) and Theorem VI.3.37 in [8, p. 318], to is easily proved that the family of processes $\left(\varepsilon \int_0^t \langle c, \nabla \widetilde{w}_i \rangle (\mathsf{X}_s^\varepsilon/\varepsilon)\,\mathrm{d}s\right)_{\varepsilon>0}$ converges in probability to the deterministic process $(t\overline{c}_i)_{t\geqslant 0}$, with $\overline{c}_i = \int_\square \langle c, \nabla \widetilde{w} \rangle (x) m(x)\,\mathrm{d}x$.

Furthermore, with the Aronson estimate, it is clear that $\mathsf{X}_{\varepsilon^2}^\varepsilon$ converges in probability to x if $\mathsf{X}_0^\varepsilon = x$ for any $\varepsilon > 0$. Putting all the previous convergence results together, if $\mathsf{X}_0^\varepsilon = x$,

$$\left(\mathsf{X}_t^\varepsilon - \frac{\beta}{\varepsilon} t\right)_{t\geqslant 0} \xrightarrow[\varepsilon\to 0]{\text{dist.}} \left(x + \overline{M}_t + t\overline{c}\right)_{t\geqslant 0},$$

where $\overline{c}$ is the vector $(\overline{c}_1, \dots, \overline{c}_N)$.

4 Homogenization with differential terms of lower order

We are now interested to prove by some probabilistic method some homogenization result for the family of operators $(A^\varepsilon)_{\varepsilon>0}$ given by $L^\varepsilon + e(\cdot/\varepsilon) + \partial_{x_i}(\widehat{b}_i(\cdot/\varepsilon)\cdot)$, where e and $\widehat{b}$ are periodic and bounded. To simplify, the constant β given by (8) is also assumed to be equal to 0.

We assume that these operators A^ε are defined on a domain $\mathcal{O}$ with a regular boundary. Let τ be the first-exit time of $\mathcal{O}$ for the process X^ε generated by L^ε.

4.1 Homogenization for a smooth differential term

If we assume that $\widehat{b}$ is smooth, then $\partial_{x_i}\widehat{b}(\cdot/\varepsilon)\cdot) = \frac{1}{\varepsilon}\operatorname{div}\widehat{b}(\cdot/\varepsilon) + \widehat{b}(\cdot/\varepsilon)\partial_{x_i}$. In [16], É. Pardoux has developped a method to deal with a highly-oscillating zero order term. This method will be employed here on some smooth functions converging to $\widehat{b}$.

So, to simplify, let us consider for a moment the homogenization problem for

$$A^\varepsilon = L^\varepsilon + e(\cdot/\varepsilon) + \frac{1}{\varepsilon} d(\cdot/\varepsilon),$$

where d is a bounded, periodic function such that $\int_\square d(x)m(x)\,\mathrm{d}x = 0$. We also assume that $\beta = 0$, where β is the constant (8). This operator A^ε generates a semi-group $(T_t^\varepsilon)_{t>0}$ may be given by the Feynman-Kac formula:

for any f in $\mathrm{L}^2(\mathcal{O})$,

$$T^\varepsilon_t f(x) = \mathbb{E}^\varepsilon_x\left[f(\mathsf{X}_t)\exp\left(\int_0^t e(\mathsf{X}_s)\,\mathrm{d}s + \frac{1}{\varepsilon}\int_0^t d(\mathsf{X}_s)\,\mathrm{d}s\right); t<\tau\right].$$

This formula is true for any x assuming that the chosen version of the semi-group $(T^\varepsilon_t)_{t>0}$ is the one given by the density transition function.

There exists a unique periodic solution $\widehat{d}\in\mathrm{H}^1(\square)$ to $\widetilde{L}\widehat{d}=-d$ satisfying $\int_\square \widehat{d}(x)m(x)\,\mathrm{d}x=0$.

Hence, if $\widehat{d}^\varepsilon(x)=\varepsilon\widehat{d}(x/\varepsilon)$, then there exists a local martingale N^ε with quadratic variation $\langle \mathsf{N}^\varepsilon\rangle_t=\int_{\varepsilon^2}^t\langle a\nabla\widehat{d}^\varepsilon,\nabla\widehat{d}^\varepsilon\rangle(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s$ such that

$$\frac{1}{\varepsilon}\int_{\varepsilon^2}^t d(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s+\int_{\varepsilon^2}^t e(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s=\mathsf{N}^\varepsilon_t+\mathsf{U}^\varepsilon_t$$

with

$$\mathsf{U}^\varepsilon_t=\int_{\varepsilon^2}^t e(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s-\widehat{d}^\varepsilon(\mathsf{X}^\varepsilon_t)+\widehat{d}^\varepsilon(\mathsf{X}^\varepsilon_{\varepsilon^2})+\int_{\varepsilon^2}^t\langle c,\nabla\widehat{d}\rangle(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s-\frac{1}{\varepsilon}\int_0^{\varepsilon^2}\widehat{d}(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s.$$

In view of the previous work, it is clear that

$$\mathsf{U}^\varepsilon \xrightarrow[\varepsilon\to0]{\text{proba}}\left(t\int_\square e(x)m(x)\,\mathrm{d}x+t\int_\square\langle c,\nabla\widehat{d}\rangle(x)m(x)\,\mathrm{d}x\right)_{t\geqslant0}=(t\overline{U})_{t\geqslant0},\qquad(11)$$

and that N^ε converges in distribution to the martingale with quadratic variations $t\overline{d}=t\int_\square\langle a\nabla\widehat{d},\nabla\widehat{d}\rangle(x)m(x)\,\mathrm{d}x$. Now, let us defined by $\widehat{\mathbb{P}}^\varepsilon_x$ the new probability measure by its Radon-Nicodym density with respect to $\mathbb{P}^\varepsilon_x$:

$$\left.\frac{\mathrm{d}\widehat{\mathbb{P}}^\varepsilon_x}{\mathrm{d}\mathbb{P}^\varepsilon_x}\right|_{\mathcal{F}^\varepsilon_t}=\exp\left(\mathsf{N}^\varepsilon_t-\frac{1}{2}\langle\mathsf{N}^\varepsilon\rangle_t\right),$$

where $\mathcal{F}^\varepsilon$ is the completion of the filtration generated by $(\mathsf{X}^\varepsilon,\mathbb{P}^\varepsilon)$. For that, some condition on N^ε has to be given, and $\widehat{\mathbb{P}}^\varepsilon_x$ is well defined for example if

$$\sup_{\varepsilon>0}\mathbb{E}^\varepsilon_x\left[\exp\left(6\langle\mathsf{N}^\varepsilon\rangle_t\right)\right]<+\infty.\qquad(12)$$

Let us assume for a moment that this condition is satisfied. Then, using the Cauchy-Schwarz estimate, the convergence in probability to some deterministic function under $\mathbb{P}^\varepsilon_x$ implies the convergence in probability to the same limit under $\widehat{\mathbb{P}}^\varepsilon_x$. Assuming that the family $(\exp(\langle\mathsf{N}^\varepsilon\rangle_t+\mathsf{U}^\varepsilon_t))_{\varepsilon>0}$ is uniformly integrable, as in [16],

$$T^\varepsilon_t f(x)\xrightarrow[\varepsilon\to0]{}\overline{T}_t f(x)\stackrel{\text{def}}{=}\mathbb{E}_x\left[f(\overline{\mathsf{X}}_t)\exp\left(t(\overline{d}+\overline{U})\right);t<\tau\right],$$

where $\overline{d} = \int_{\square}\langle a\nabla\widehat{d}, \nabla\widehat{d}\rangle(x)m(x)\,\mathrm{d}x$. The term $\overline{U}$ is given by (11), and $\overline{\mathsf{X}}$ is the process whose infinitesimal generator is

$$\frac{1}{2}\overline{a}_{i,j}\frac{\partial^2}{\partial x_i\partial x_j} + (\overline{c}+\overline{D})\frac{\partial}{\partial x_i},\ \text{with } \overline{D} = \int_{\square}\langle a\nabla\widehat{d}, \nabla\widetilde{w}_i\rangle(x)m(x)\,\mathrm{d}x.$$

The term $\overline{D}$ comes from the Girsanov transform corresponding to the change of measure from $\mathbb{P}^{\varepsilon}_x$ to $\widehat{\mathbb{P}}^{\varepsilon}_x$.

Now it remains to prove (12) and a condition of uniform integrability. For that, it is sufficient to prove that

$$\sup_{\varepsilon>0}\mathbb{E}\left[\exp\left(\int_{\varepsilon^2}^t g(\mathsf{X}^{\varepsilon}_s/\varepsilon)\,\mathrm{d}s\right)\right] < +\infty, \tag{13}$$

for $g = \langle c, \nabla\widehat{d}\rangle$ and $g = \langle a\nabla\widehat{d}, \widehat{d}\rangle$. But is is important to note that, unless a is weakly differentiable, the term $\nabla\widehat{d}$ is in general *not* bounded, so that we have to prove (13) for g only in $\mathrm{L}^1(\square)$. This is a supplementary difficulty with respect to [16] and the use of non-divergence form operators.

Proposition 3. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of non-negative functions in $\mathrm{L}^1(\square)$ converging in $\mathrm{L}^1(\square)$ to some non-negative function g. Then, if $(n_\varepsilon)_{\varepsilon>0}$ is a sequence of integers converging to infinity fast enough, there exist some sequences $(\alpha_\varepsilon)_{\varepsilon>0}$ and $(\gamma_\varepsilon)_{\varepsilon>0}$ such that for any $t \geqslant 0$,*

$$\mathbb{E}_x\left[\exp\left(\int_{\varepsilon^2}^t g_{n_\varepsilon}(\mathsf{X}^{\varepsilon}_s/\varepsilon)\,\mathrm{d}s\right)\right] \leqslant e^{\alpha_\varepsilon t+\gamma_\varepsilon}, \tag{14}$$

$$\textit{with } \alpha_\varepsilon \xrightarrow[\varepsilon\to 0]{} \int_{\square} g(x)m(x)\,\mathrm{d}x \textit{ and } \gamma_\varepsilon \xrightarrow[\varepsilon\to 0]{} 0.$$

Proof. Our proof is inspired by that of the Kas'minskii Lemma on functions in the Kato class (see for example [20, Lemma 2.1 and Theorem 2.2, p. 8]).

Using a generalization of the proof of Corollary 1, for any integer $k \geqslant 1$,

$$\begin{aligned}
&\mathbb{E}_x\left[\left(\int_{\varepsilon^2}^t g_{n_\varepsilon}(\mathsf{X}^{\varepsilon}_s/\varepsilon)\,\mathrm{d}s\right)^k\right] \\
&\qquad = k!\int_{0\leqslant s_1\leqslant s_2\leqslant\cdots\leqslant s_{k-1}\leqslant t-\varepsilon^2}\mathbb{E}_x\Bigg[g_{n_\varepsilon}(\mathsf{X}^{\varepsilon}_{s_1+\varepsilon^2}/\varepsilon)\cdots g_{n_\varepsilon}(\mathsf{X}^{\varepsilon}_{s_{k-1}+\varepsilon^2}/\varepsilon) \\
&\qquad\qquad\times\mathbb{E}_{\mathsf{X}^{\varepsilon}_{s_{k-1}+\varepsilon^2}}\left[\int_{\varepsilon^2}^{t-s_{k-1}} g_{n_\varepsilon}(\mathsf{X}^{\varepsilon}_s/\varepsilon)\,\mathrm{d}s\right]\Bigg]\,\mathrm{d}s_1\cdots\mathrm{d}s_{k-1},
\end{aligned} \tag{15}$$

and, with (6) in Proposition 2,

$$\begin{aligned}
\mathbb{E}_{\mathsf{X}^{\varepsilon}_{s_{k-1}+\varepsilon^2}}\left[\int_{\varepsilon^2}^{t-s_{k-1}} g_{n_\varepsilon}(\mathsf{X}^{\varepsilon}_s/\varepsilon)\,\mathrm{d}s\right] &\leqslant \sup_{x\in\mathbb{R}^N}\varepsilon^2\int_1^{t/\varepsilon^2} {}^{\varepsilon}Q_s g_{n_\varepsilon}(x)\,\mathrm{d}x \\
&\leqslant t\overline{g} + \varepsilon^2 K\,\|g\|_{\mathrm{L}^1(\square)} + C\,\|g_{n_\varepsilon}-g\|_{\mathrm{L}^1(\square)}
\end{aligned}$$

with $\overline{g} = \fint_{\square} g(x)m(x)\,\mathrm{d}x$. So, if $(n_\varepsilon)_{\varepsilon>0}$ is choose so that $\|g_{n_\varepsilon} - g\|_{\mathrm{L}^1(\square)} = \mathrm{o}(\varepsilon)$, an induction leads to

$$\mathbb{E}_x\left[\left(\int_{\varepsilon^2}^t g_{n_\varepsilon}(\mathsf{X}_s^\varepsilon/\varepsilon)\,\mathrm{d}s\right)^k\right] \leqslant k!(t\overline{g}+\mathrm{o}(\varepsilon))^k$$

$$\text{and}\ \sup_{x\in\mathbb{R}^N} \mathbb{E}_x\left[\exp\left(\int_{\varepsilon^2}^t g_{n_\varepsilon}(\mathsf{X}_s^\varepsilon/\varepsilon)\,\mathrm{d}s\right)\right] \leqslant \sum_{k\geqslant 0}(t\overline{g}+\mathrm{o}(\varepsilon))^k.$$

The previous sum is equal to $1/(1-t\overline{g}+\mathrm{o}(\varepsilon))$ if $t\overline{g}+\mathrm{o}(\varepsilon)$ is smaller than 1. Now, let us fix ε and set $t_0 = \varepsilon/\overline{g}$. If k is an integer such that $t = kt_0 + r$ with $0 \leqslant r < t_0$, then the Markov property implies again that

$$\begin{aligned}\mathbb{E}_x\left[\exp\left(\int_{\varepsilon^2}^t g_{n_\varepsilon}(\mathsf{X}_s^\varepsilon/\varepsilon)\,\mathrm{d}s\right)\right] &\leqslant \mathbb{E}_x\Bigg[\exp\left(\int_0^{k(t_0-\varepsilon^2)} g_{n_\varepsilon}(\mathsf{X}_{s+\varepsilon^2}^\varepsilon/\varepsilon)\,\mathrm{d}s\right) \\ &\quad\times \mathbb{E}_{\mathsf{X}_{kt_0}^\varepsilon}\left[\exp\left(\int_0^{t_0-\varepsilon^2} g_{n_\varepsilon}(\mathsf{X}_{s+\varepsilon^2}^\varepsilon/\varepsilon)\,\mathrm{d}s\right)\right]\Bigg] \\ &\leqslant \left(\frac{1}{1-\varepsilon+\mathrm{o}(\varepsilon)}\right)^{k+1} \leqslant \frac{1}{1-\varepsilon+\mathrm{o}(\varepsilon)}\exp\left(\frac{\overline{g}t}{\varepsilon}\log\left(\frac{1}{1-\varepsilon+\mathrm{o}(\varepsilon)}\right)\right),\end{aligned}$$

if ε is smaller than some ε_0, and the result is proved. ∎

The proof is specific to the periodic case, and is not valid for a general random media, for which the uniform integrability shall be proved using another strategy [9].

Proposition 3 also gives us a control as ε is small on the maximal growth of the solution of the parabolic PDE.

Corollary 2. *There exist two sequences $(\alpha_\varepsilon)_{\varepsilon>0}$ and $(\gamma_\varepsilon)_{\varepsilon>0}$ satisfying $\gamma_\varepsilon \xrightarrow[\varepsilon\to 0]{} 0$ and*

$$\alpha_\varepsilon \xrightarrow[\varepsilon\to 0]{} \frac{1}{2}\int_\square \langle a\nabla\widehat{d}, \nabla\widehat{d}\rangle(x)m(x)\,\mathrm{d}x + \int_\square \langle c, \nabla\widehat{d}\rangle(x)m(x)\,\mathrm{d}x = \overline{\alpha}$$

for which

$$|T^\varepsilon f(x)| \leqslant \|f\|_\infty \exp\left(\alpha_\varepsilon t + \|e\|_\infty t + \gamma_\varepsilon\right) \tag{16}$$

for any measurable, bounded function f and any starting point x.

Proof. We remark first that

$$\mathbb{E}_x^\varepsilon\left[\exp\left(\mathsf{N}_t^\varepsilon + \mathsf{U}_t^\varepsilon\right) f(\mathsf{X}_t)\right] \leqslant \widehat{\mathbb{E}}_x^\varepsilon\left[\exp\left(\frac{1}{2}\langle \mathsf{N}^\varepsilon\rangle_t + \mathsf{U}_t^\varepsilon\right)\right]\|f\|_\infty\,.$$

Except $\langle \mathsf{N}^\varepsilon\rangle_t$ and $\int_{\varepsilon^2}^t \langle c, \nabla\widehat{d}\rangle(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s$, the other terms that appear in U^ε_t are smaller than $\|e\|_\infty \cdot t + \mathrm{o}(\varepsilon)$ for ε small enough. Proposition 3 asserts the existence of the sequences $(\alpha_\varepsilon)_{\varepsilon>0}$ and $(\gamma_\varepsilon)_{\varepsilon>0}$, and for any x,

$$\mathbb{E}_x\left[\exp\left(\frac{1}{2}\langle \mathsf{N}^\varepsilon\rangle_t + \int_{\varepsilon^2}^t \langle c, \nabla\widehat{d}\rangle(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s\right)\right] \leqslant \exp(\alpha_\varepsilon t + \gamma_\varepsilon).$$

The result is then proved. ∎

The previous Corollary gives some indication on the spectrum of A^ε. For example, if $\mathcal{O}$ is a bounded open set with smooth boundary, the maximum principle implies that any eigenfunction associated to a real eigenvalue of A^ε is bounded. So, the maximal eigenvalue of A^ε is bounded in ε. Then, $\alpha - A^\varepsilon$ is invertible for α for some large enough independent of ε.

4.2 Passage to a non-smooth term

Now, we do not assume that $\widehat{b}$ is smooth, but we shall assume that $\int_\square \widehat{b}(x)\nabla m(x)\,\mathrm{d}x = 0$. We also assume that $b = 0$, and we set

$$A^\varepsilon = L^\varepsilon + e(\cdot/\varepsilon) + \frac{\partial}{\partial x_i}(\widehat{b}_i(\cdot/\varepsilon)\cdot).$$

Proposition 4. *Let $\mathcal{O}$ be an open domain with smooth boundary. Then, for any α large enough and for any f in $\mathrm{L}^2(\mathcal{O})$, the solution of the elliptic equation*

$$(\alpha - A^\varepsilon)v^\varepsilon = f \text{ and } v^\varepsilon \in \mathrm{H}^1_0(\mathcal{O}),$$

converges pointwise as ε goes to 0 to the solution of

$$(\alpha - \overline{A})v = f \text{ and } v \in \mathrm{H}^1_0(\mathcal{O}),$$

where $\overline{A}$ is a second-order differential operator

$$\overline{A} = \overline{L} + \int_\square (e + \langle c, \nabla\widehat{u}\rangle)\,\mathrm{d}x$$

$$\text{with } \overline{L} = \frac{1}{2}\overline{a}_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \left(\int_\square \left(\langle c, \nabla\widetilde{w}_i\rangle + \langle \widehat{b}, \nabla\widetilde{w}_i\rangle\right)\mathrm{d}x\right)\frac{\partial}{\partial x_i}$$

$$\text{and } \widehat{u} \in \mathrm{H}^1(\square) \text{ is solution to } \widetilde{L}\widehat{u} = -\operatorname{div}\widehat{b} \text{ and } \int_\square \widehat{u}(x)m(x)\,\mathrm{d}x = 0.$$

A non-null Dirichlet boundary condition (*i.e.*, $v(x) = \varphi(x)$ on the boundary of $\mathcal{O}$ for a continuous function φ) may also be considered, using the probabilistic representation for such a problem (see [3] for a proof of such a result. The representation is the same as for non-divergence form operators). This also implies the convergence of the solutions of the parabolic PDE involving A^ε, even if $\mathcal{O}$ is not bounded.

Proof. Let $(\widehat{b}_n)_{n\in\mathbb{N}}$ be a family of measurable, periodic functions bounded by Λ such that

$$\widehat{b}_n \text{ is smooth, } \widehat{b}_n \xrightarrow[n\to+\infty]{\mathrm{L}^2(\square)} \widehat{b} \text{ and } \int_\square \widehat{b}_n(x)\nabla m(x)\,\mathrm{d}x = 0.$$

Let us denote by $\widehat{u}_n$ the solutions of $\widetilde{L}\widehat{u}_n = -\operatorname{div}\widehat{b}_n$ with $\int_\square \widehat{u}_n(x)m(x)\,\mathrm{d}x = 0$. Using (5), and the Poincaré inequality, there exists some constant C depending only on λ and Λ such that $\|\widehat{u}-\widehat{u}_n\|_{\mathrm{H}^1(\square)} \leqslant C\|\widehat{b}-\widehat{b}_n\|_{\mathrm{L}^2(\square)}$. Hence $(\widehat{u}_n)_{n\in\mathbb{N}}$ converges to $\widehat{u}$ in $\mathrm{H}^1(\square)$.

Let us denote by $(T_t^\varepsilon)_{t>0}$ the semi-group whose infinitesimal generator is A^ε and $(T_t^{\varepsilon,n_\varepsilon})_{t>0}$ that of generator $A^{\varepsilon,n_\varepsilon}$ given by

$$\frac{1}{2}\frac{\partial}{\partial x_i}\left(a_{i,j}(\cdot/\varepsilon)\frac{\partial}{\partial x_j}\right) + c_i(\cdot/\varepsilon)\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i}(\widehat{b}_{n_\varepsilon}(\cdot/\varepsilon)\cdot) + e(\cdot/\varepsilon).$$

for some fixed sequence $(n_\varepsilon)_{\varepsilon>0}$ that increases to infinity. Since $\widehat{b}_{n_\varepsilon}$ is smooth, it is clear that

$$\frac{\partial}{\partial x_i}(\widehat{b}_{n_\varepsilon}(\cdot/\varepsilon)\cdot) = \frac{1}{\varepsilon}\operatorname{div}\widehat{b}_{n_\varepsilon}(\cdot/\varepsilon) + \widehat{b}_{n_\varepsilon}(\cdot/\varepsilon)\frac{\partial}{\partial x_i}.$$

Let f be a smooth function with compact support in $\mathcal{O}$, As $\nabla\widehat{u}_{n_\varepsilon}$ converges in $\mathrm{L}^2(\square)$ to $\nabla\widehat{u}$, it follows from the results in the previous section that $T_t^{\varepsilon,n_\varepsilon}f(x) \xrightarrow[\varepsilon\to 0]{} \overline{T}_t f(x)$, where $(\overline{T}_t)_{t>0}$ is the semi-group generated by $\overline{A}$. We have to note that some terms which appear in the previous Section have been cancelled with some integration by part.

It remains to identifies the limit: On a bounded domain $\mathcal{O}$, let us consider the two problems

$$\begin{cases}(\alpha - A^{\varepsilon,n_\varepsilon})u^\varepsilon = f \text{ on } \mathcal{O},\\ u\in\mathrm{H}_0^1(\mathcal{O}),\end{cases} \quad\text{and}\quad \begin{cases}(\alpha - A^\varepsilon)v^\varepsilon = f \text{ on } \mathcal{O},\\ v\in\mathrm{H}_0^1(\mathcal{O})\end{cases}$$

where f is a continuous function with compact support on $\mathcal{O}$. The solution u^ε is equal to $u^\varepsilon(x) = \int_0^{+\infty} e^{-\alpha t}T_t^{\varepsilon,n_\varepsilon}f(x)\,\mathrm{d}t$ and converges to the solution $v\in\mathrm{H}_0^1(\mathcal{O})$ of the elliptic problem $(\alpha-\overline{A})v = f$.

In the case of $b=0$, then for α large enough the family of operators L^ε are coercive for the $\mathrm{H}_0^1(\mathcal{O})$-norm independently of ε. From Corollary 2, u^ε is bounded uniformly in ε. Classical estimates imply that $\|v^\varepsilon - u^\varepsilon\|_{\mathrm{L}^2(\mathcal{O}}$ converges to 0. Hence, v^ε converges in $\mathrm{L}^2(\mathcal{O})$ to v. In this case, as the coefficients of the involved differential operators A^ε are bounded by some constant independent of ε, it is possible to use extensively the Aronson estimate (2) to prove that the convergences hold pointwise for any function f in $\mathrm{L}^2(\mathcal{O})$ (see [17]). ∎

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